SOME PROBLEMS IN THE THEORY OF ELASTICITY OF NONHOMOGENEOUS ELASTIC MEDIA

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Media are considered where elastic properties vary from point to point. The variation of elastic properties of the medium may occur continuously or abruptly, whereby the continuous variation might be in the form of discrete steps. The latter is the case in synthetic laminated materials; within each layer there is a continuous variation of elastic properties which terminates in an abrupt jump on the boundaries of the layers.

In the following, one considers merely the continuous nonhomogeneity of the medium which corresponds to the classic concept of the medium in the conventional theory of elasticity of homogeneous media.

Continuous nonhomogeneity may be either isotropic or anisotropic. The isotropic nonhomogeneity of a medium will be understood to characterize a body in which the elastic modulus and the Poisson ratio may vary from point to point; however, the number of independent functions determining the elastic properties, as always, equals two. Also, if at a selected point one chooses an arbitrary direction, the elastic properties are equal in all the directions and there are no preferred structural orientations.

Anisotropic nonhomogeneity of a body may be of a twofold character. Firstly, nonhomogeneity manifested merely by the change of magnitude of the elastic properties from point to point irrespective of the orientation of the coordinate axes.

The second type of anisotropic nonhomogeneity is such that the orientation of the principal axes of anisotropy is different at various points of the body whereby these orientations vary continuously from point to point; the magnitudes of elastic properties of the body also vary continuously. This type of anisotropy is the most general one.

1. Axisymmetric problem in theory of elasticity of nonhomogeneous isotropic media. We shall introduce the following notations: G^* is a variable shear modulus dependent on the coordinates of the point; E^* is a variable longitudinal modulus of elasticity; ν^* is a variable Poisson ratio; λ^* is a variable Lamé modulus; θ is volumetric change. Let ξ be the component of displacement in radial direction and ζ the component of displacement along the axis z, which is assumed to be the axis of symmetry of the problem. The relation between the components of the stress and strain tensors retains its ordinary form also in this case of a nonhomogeneous medium

$$\sigma_r = 2G^* \frac{\partial \xi}{\partial r} + \lambda^*\theta, \qquad \sigma_z = 2G^* \frac{\partial \zeta}{\partial z} + \lambda^*\theta \tag{1.1}$$

$$\sigma_\vartheta = 2G^* \frac{\xi}{r} + \lambda^*\theta, \qquad \tau_{rz} = G^* \left(\frac{\partial \zeta}{\partial r} + \frac{\partial \xi}{\partial r}\right)$$

$$\theta = \frac{\partial \xi}{\partial r} + \frac{\xi}{r} + \frac{\partial \zeta}{\partial z} \tag{1.2}$$

Substituting Expressions (1.1) into the equilibrium equations

$$\frac{\partial \sigma_z}{\partial r} + \frac{\partial \tau_{rz}}{\partial z} + \frac{\sigma_r - \sigma_\vartheta}{r} = 0, \qquad \frac{\partial \tau_{zr}}{\partial r} + \frac{\partial \sigma_z}{\partial z} + \frac{\tau_{zr}}{r} = 0 \tag{1.3}$$

we obtain a system of differential equations with regard to ξ and ζ

$$G^* \left(\triangle \xi + \frac{\partial \theta}{\partial r} - \frac{\xi}{r^2}\right) + 2\frac{\partial G^*}{\partial r}\frac{\partial \xi}{\partial r} + \frac{\partial G^*}{\partial z}\left(\frac{\partial \zeta}{\partial r} + \frac{\partial \xi}{\partial r}\right) + \frac{\partial}{\partial r}(\lambda^*\theta) = 0$$

$$G^* \left(\triangle \zeta + \frac{\partial \theta}{\partial z}\right) + 2\frac{\partial G^*}{\partial z}\frac{\partial \zeta}{\partial z} + \frac{\partial G^*}{\partial r}\left(\frac{\partial \zeta}{\partial r} + \frac{\partial \xi}{\partial z}\right) + \frac{\partial}{\partial z}(\lambda^*\theta) = 0 \tag{1.4}$$

In order to render the problem considered here meaningful it is necessary that two arbitrary elastic characteristics of the nonhomogeneous medium be given as axisymmetric functions of coordinates. Let us assume the following functions:

$$G^* = Ge^{\alpha z}, \qquad \nu^* = \text{const} = \nu \tag{1.5}$$

where G and α are constants. The quantity α may be positive or negative.

We introduce an auxiliary function by means of a relation

$$\xi = \partial \varkappa / \partial r \tag{1.6}$$

Eliminating function ζ from the system (1.4), in view of the assumed nonhomogeneity (1.5), we obtain the following equation for the function κ:

$$\triangle\triangle\varkappa - a\triangle\varkappa + 2\alpha\frac{\partial}{\partial z}\triangle\varkappa + b\frac{\partial^2 \varkappa}{\partial z^2} = 0 \qquad \left(a = \frac{\alpha^2 \nu}{1-\nu},\ b = \frac{\alpha^2}{1-\nu}\right) \tag{1.7}$$

We may observe that for $\alpha = 0$ the differential equation (1.7) becomes a biharmonic equation and the function κ becomes in this case the Love function.

In the axisymmetric case the most interesting are two classes of problems: 1) semi-infinite space loaded on the limiting surface by a prescribed set of surface forces;;2) circular cylinder with a prescribed set of surface forces. In both cases, in order to satisfy the boundary conditions the solution of Equation (1.7) should have the following form:

$$\varkappa(r, z) = e^{mz}\Phi(r) \qquad (1.8)$$

where m is as yet an arbitrary complex quantity. Substitution of (1.8) into (1.7) gives the following equation for the function $\Phi(r)$:

$$\frac{d^4\Phi}{dr^4} + \frac{2}{r}\frac{d^3\Phi}{dr^3} + \left(k^2 - \frac{1}{r^2}\right)\frac{d^2\Phi}{dr^2} + \left(\frac{k^2}{r} + \frac{1}{r^3}\right)\frac{d\Phi}{dr} + n^4\Phi = 0 \qquad (1.9)$$

Here

$$k^2 = 2m(m + \alpha) - a, \qquad n^4 = m^2(m + \alpha)^2 \qquad (1.10)$$

Equation (1.9) may be put in the form

$$\left[\frac{d^2(\;)}{dr^2} + \frac{1}{r}\frac{d(\;)}{dr} + q_1^2(\;)\right]\left[\frac{d^2(\;)}{dr^2} + \frac{1}{r}\frac{d(\;)}{dr} + q_2^2(\;)\right]\Phi = 0 \qquad (1.11)$$

Here

$$q_{1,2}^2 = \frac{k^2}{2} \pm \sqrt{\frac{k^4}{4} - n^4} \qquad (1.12)$$

Differential equation (1.11) splits into two separate equations

$$\frac{d^2\Phi_1}{dr^2} + \frac{1}{r}\frac{d\Phi_1}{dr} + q_1^2\Phi_1 = 0, \qquad \frac{d^2\Phi_2}{dr^2} + \frac{1}{r}\frac{d\Phi_2}{dr} + q_2^2\Phi_2 = 0 \qquad (1.13)$$

Each of these equations can be reduced to the equation of the Bessel type or a modified Bessel type depending on the signs of q_1^2 and q_2^2.

For the problems of the class I, i.e. for a semi-infinite space, it is necessary for the solution to be a Bessel function with a real argument.

This circumstance defines the range of the parameter m entering Expression (1.8), through which parameter the quantities q_1^2 and q_2^2 are determined in accordance with (1.12) and (1.10).

For the problems of class II there are other requirements resulting from the boundary conditions on the side surfaces of the cylinder. These limitations will be discussed in greater detail later in the text.

2. Nonhomogeneous semi-infinite space under a load distributed over a circular area. The exponential parameter m entering into Formula (1.8) is complex:

$$m = t + is \tag{2.1}$$

Solution of Equations (1.13) will be in the form of Bessel and Neuman functions having a real argument if q_1^2 and q_2^2 are definite positive quantities. We substitute (2.1) into (1.10) and the resulting expression into (1.12), then from the aforementioned condition we obtain

$$s = \pm \sqrt{\frac{at(t+\alpha)}{a+(2t+\alpha)^2}}, \qquad m_{1,2} = t \pm i \sqrt{\frac{at(t+\alpha)}{a+(2t+\alpha)^2}} \tag{2.2}$$

From that follows that the range of variable t is

$$0 < t < \infty \quad \text{for } \alpha > 0, \qquad -\infty < t < 0 \quad \text{for } \alpha < 0$$

Substituting Expression (2.2) for m into (1.12), we obtain

$$q = (2t+\alpha) \sqrt{\frac{t(t+\alpha)}{a+(2t+\alpha)^2}} \tag{2.3}$$

It is assumed here that $q_1^2 = q_2^2$, inasmuch as these quantities in condition (2.2) differ from each other merely by a fixed constant. We introduce the notation

$$p = \sqrt{\frac{at(t+\alpha)}{a+(2t+\alpha)^2}} \tag{2.4}$$

The general integral of Equation (1.7) in terms of Bessel and Neuman functions of a real argument has the form

$$\varkappa = \int_0^\infty e^{tz} \{[F_1(t)\cos pz + F_2(t)\sin pz]\, J_0(qr) + [F_3(t)\cos pz + F_4(t)\sin pz]\, N_0(qr)\}\, dt \tag{2.5}$$

where $F_1(t)$, $F_2(t)$, $F_3(t)$ and $F_4(t)$ are arbitrary functions. Substituting (2.5) into (1.6) and then using differential equation (1.4), we find both components of the displacement vector, the expressions for which can be given as

$$\xi = \frac{\partial \varkappa}{\partial r} = -\int_0^\infty e^{tz} q\, [F_1(t)\cos pz + F_2(t)\sin pz]\, J_1(qr)\, dt$$
$$\zeta = \int_0^\infty e^{tz} [\psi_1(r,t)\cos pz + \Psi_2(r,t)\sin pz]\, dt \tag{2.6}$$

In obtaining (2.6) the part of solution containing Neuman functions has been neglected since the latter are not relevant for the case of a semi-infinite space. Also the following notations have been adopted:

$$\Psi_1(r,t) = \frac{1}{u^2+p^2}[u\Phi_1(r,t) + p\Phi_2(r,t)] \qquad (u = t + \alpha(1-2\nu))$$

$$\Psi_2(r,t) = \frac{1}{u^2+p^2}[p\Phi_1(r,t) + u\Phi_2(r,t)]$$

$$\Phi_1(r,t) = [q^2F_1(t) - (1-2\nu)\sqrt{a}qF_2(t)]J_0(qr)$$

$$\Phi_2(r,t) = [q^2F_2(t) + (1-2\nu)\sqrt{a}qF_1(t)]J_0(qr) \qquad (2.7)$$

Components of the stress tensor may be found from Expressions (1.1) after the substitution therein of Expressions (1.5), (2.6) and the above-defined abbreviations (2.7)

$$\begin{aligned}\sigma_z = \frac{2Ge^{\alpha z}}{1-2\nu}\int_0^\infty e^{tz}J_0(qr)\Big\{F_1(t)\Big[\nu(-t\alpha\cos pz - 2tp\sin pz -\\ - t^2\cos pz + 2tp\sin pz + p^2\cos pz) + \frac{(1-\nu)}{u^2+p^2}\cos pz\,(tq^2u +\\ + tqp\sqrt{a}(1-2\nu) + q^2p^2 + qp\sqrt{a}u(1-2\nu)) +\\ + \frac{(1-\nu)}{u^2+p^2}\sin pz\,(q^2pt + q\sqrt{a}tu(1-2\nu) - q^2pu - q\sqrt{a}p^2(1-2\nu))\Big] +\\ + F_2(t)\Big[\nu(-t\alpha\sin pz + 2tp\cos pz - t^2\sin pz - 2tp\cos pz +\\ + p^2\sin pz) + \frac{(1-\nu)}{u^2+p^2}\cos pz\,(q^2tp - q\sqrt{a}tu(1-2\nu) +\\ + q^2pu - p^2q\sqrt{a}(1-2\nu)) + \frac{(1-\nu)}{u^2+p^2}\sin pz\,(tq^2u -\\ - tqp\sqrt{a}(1-2\nu) - q^2p^2 + qp\sqrt{a}u(1-2\nu))\Big]\Big\}dt\end{aligned} \qquad (2.8)$$

$$\begin{aligned}\tau_{rz} = -Ge^{\alpha z}\int_0^\infty e^{tz}qJ_1(qr)\Big\{F_1(t)\Big[\cos pz\Big[\frac{1}{u^2+p^2}(q^2u +\\ + qp\sqrt{a}(1-2\nu)) + t\Big] + \sin pz\Big[\frac{1}{u^2+p^2}(q^2p + qu\sqrt{a}(1-2\nu)) - p\Big]\Big] +\\ + F_2(t)\Big[\cos pz\Big[\frac{1}{u^2+p^2}(q^2p - qu\sqrt{a}(1-2\nu)) + p\Big] +\\ + \sin pz\Big[\frac{1}{u^2+p^2}(q^2u - qp\sqrt{a}(1-2\nu)) + t\Big]\Big]\Big\}dt\end{aligned} \qquad (2.9)$$

$$\begin{aligned}\sigma_r = 2Ge^{\alpha z}\Big\{-\int_0^\infty e^{tz}q^2[F_1(t)\cos pz + F_2(t)\sin pz]\frac{dJ_1(qr)}{d(qr)}dt +\\ + \frac{\nu}{1-2\nu}\int_0^\infty e^{tz}J_0(qr)\Big\{F_1(t)\Big[\cos pz\Big[-q^2 + \frac{1}{u^2+p^2}(q^2tu +\end{aligned}$$

$$+ qp\sqrt{a}\, t\,(1-2\nu) + q^2p^2 + qp\sqrt{a}\,(1-2\nu)\,u)\Big] +$$

$$+ \frac{\sin pz}{u^2+p^2}\,[q\sqrt{a}\,tu\,(1-2\nu) - q^2p\alpha\,(1-2\nu) - qp^2\sqrt{a}\,(1-2\nu)]\Big] +$$

$$+ F_2(t)\left[\frac{\cos pz}{u^2+p^2}\,[2q^2pt - q\sqrt{a}\,tu\,(1-2\nu) + q^2p\alpha\,(1-2\nu) -\right.$$

$$- qp^2\sqrt{a}\,(1-2\nu)] + \sin pz\Big[-q^2 + \frac{1}{u^2+p^2}\,(tq^2u - qp\sqrt{a}\,t\,(1-2\nu) -$$

$$\left.- q^2p^2 + qp\sqrt{a}\,u\,(1-2\nu))]\Big]\Big]\Big\}\,dt\Big\} \tag{2.10}$$

$$\sigma_\vartheta = 2Ge^{\alpha z}\Big\{-\frac{1}{r}\int_0^\infty e^{tz}qJ_1(qr)\,[F_1(t)\cos pz + F_2(t)\sin pz]\,dt +$$

$$+ \frac{\nu}{1-2\nu}\int_0^\infty e^{tz}J_0(qr)\Big\{F_1(t)\Big[\cos pz\Big[-q^2 + \frac{1}{u^2+p^2}\,(q^2tu + qp\sqrt{a}\,t\,(1-2\nu) +$$

$$+ q^2p^2 + qp\sqrt{a}\,u\,(1-2\nu))\Big] + \frac{\sin pz}{u^2+p^2}\,[q\sqrt{a}\,tu\,(1-2\nu) - q^2p\alpha\,(1-2\nu) -$$

$$- qp^2\sqrt{a}\,(1-2\nu)]] + F_2(t)\Big[\frac{\cos pz}{u^2+p^2}\,[2q^2pt - q\sqrt{a}\,tu\,(1-2\nu) +$$

$$+ q^2p\alpha\,(1-2\nu) - qp^2\sqrt{a}\,(1-2\nu)] + \sin pz\Big[-q^2 +$$

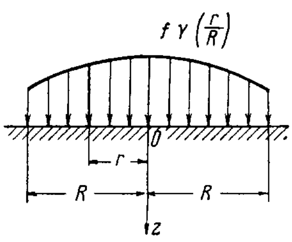
$$\frac{1}{u^2+p^2}\,(tq^2u - qp\sqrt{a}\,t\,(1-2\nu) - q^2p^2 + qp\sqrt{a}\,u\,(1-2\nu)\Big]\Big]\Big\}\,dt \tag{2.11}$$

Functions $F_1(t)$ and $F_2(t)$ entering into the above expressions are determined from the boundary conditions.

Let us consider the following boundary conditions for $z = 0$:

$$\tau_{rz} = 0 \quad \text{for } 0 < r < \infty, \quad \sigma_z = \begin{cases} -f\gamma\,(r/R) & \text{for } r \leqslant R \\ 0 & \text{for } r > R \end{cases} \tag{2.12}$$

Here R is the radius of the circle at which the load is applied to the semi-infinite space (Figure); f is a parameter characterizing the intensity of the applied load; $\gamma(r/R)$ is a function characterizing the distribution of the load.



Expressions for components τ_{rz} and σ_z using two arbitrary functions equivalent to $F_1(t)$ and $F_2(t)$ can be given in the form of Fourier and Bessel integrals as follows:

$$\tau_{rz} = Ge^{\alpha z}\int_0^\infty qJ_1(qr)\,dq\int_0^\infty s\tau\,(z,s)\,J_1(qs)\,ds \tag{2.13}$$

$$\sigma_z = \frac{2Ge^{\alpha z}}{1-2\nu}\int_0^\infty qJ_0(qr)\,dq\int_0^\infty s\sigma(z,s)\,J_0(q,s)\,ds \qquad (2.14)$$

Here $\tau(z, s)$ and $\sigma(z, s)$ are arbitrary functions of the argument s, to the same extent as the above-mentioned functions are of argument t. The connection between these two can be readily established.

Functions $\tau(z, s)$ and $\sigma(z, s)$ for $z = 0$ are determined from boundary conditions on the basis of the properties of the Fourier-Bessel integral. Determining, then, functions $F_1(t)$ and $F_2(t)$ we obtain

$$F_1(t) = -\frac{(1-2\nu)}{2G}\frac{f}{q}S(qR)\frac{Q(t)}{S(t)-P(t)\,T(t)} \qquad (2.15)$$

$$F_2(t) = -P(t)\,F_1(t)$$

Here

$$S(qR) = \int_0^{(qR)}(qs)\,\gamma\left(\frac{qs}{qR}\right)J_0(qs)\,d(qs)$$

$$Q(t) = 2\sqrt{\frac{t(t+\alpha)}{a+(2t+a)^2}} + \frac{(2t+\alpha)^2(a+\alpha^2)}{2\sqrt{(t^2+\alpha t)\,[a+(2t+\alpha^2)]^3}}$$

$$P(t) = \frac{q^2u + qp\sqrt{a}\,(1-2\nu) + t(u^2+p^2)}{q^2p - q\sqrt{a}\,u(1-2\nu) + p(u^2+p^2)} \qquad (2.16)$$

$$S(t) = \frac{1}{u^2+p^2}[-\nu q^2(u^2+p^2) + (1-\nu)\,tq^2u + (1-\nu)(1-2\nu)\,tpq\sqrt{a} + (1-\nu)\,q^2p^2 + (1-\nu)(1-2\nu)\,qp\sqrt{a}\,u]$$

$$T(t) = \frac{1}{u^2+p^2}[2q^2pt(1-\nu) - qt\sqrt{a}\,u(1-\nu)(1-2\nu) + (1-\nu)(1-2\nu)\,q^2p\alpha - (1-\nu)(1-2\nu)\,qp^2\sqrt{a}]$$

Having determined functions $F_1(t)$ and $F_2(t)$ in accordance with (2.15), and utilizing abbreviations (2.16), it is possible to compute all the components of the stress tensor from Expressions (2.8), (2.9), (2.10) and (2.11).

Let us consider a uniformly distributed load applied to a circular area of radius R. In that case the function $\gamma(r/R) = 1$. The integral $s(qR) = (qR)J_1(qR)$. Substituting into the first of Expressions (2.15), we find

$$F_1(t) = -\frac{(1-2\nu)}{2G}fRJ_1(qR)\frac{Q(t)}{S(t)-P(t)\,T(t)} \qquad (2.17)$$

All the remaining formulas obtained in the previous section remain unaltered.

Solution for the case of a load uniformly distributed over a circular area is readily obtained from the previous solution if one considers the applied circular load as a result of superposition of uniform loads distributed over a circular area and oriented in different directions.

3. Nonhomogeneous cylinder. Let us consider now the class II of problems. We consider a nonhomogeneous cylinder extending infinitely from one of its ends and also a nonhomogeneous cylinder limited on both ends. The cylinder may be assumed to be filled fully with a material, or the central coaxial part may be assumed hollow.

In the case of an infinite cylinder, the solution should be in the form of a Fourier integral and in the case of a finite cylinder it should be in the form of Fourier series.

These forms of solutions are compatible with (1.8) if the exponential also satisfies other criteria.

In order to satisfy the boundary conditions on the side wall of the cylinder, the exponential term should comprise a varying parameter with regard to which one accomplishes the integration or summation in such a way that it enters merely under the sign of trigonometric function. It is not difficult to see that the solution previously obtained does not meet this condition, since this parameter enters not only into the trigonometric function, but also into the exponential function. The exponent should therefore have the following form:

$$m = m_0 + is \tag{3.1}$$

Here m_0 is a constant, s is a varying quantity which in the following will constitute a summation or integration parameter. In addition to the above condition it is also necessary to subject (3.1) to the requirement that $q_{1,2}^2$ be negative. The latter leads to equations

$$m_0 = -\frac{\alpha}{2}, \qquad q^2 = -\left\{\left(\frac{a}{2} + \frac{\alpha}{4} + s^2\right) + \left[a\left(\frac{a}{2} + \frac{\alpha^2}{4} + s^2\right) - \frac{a^2}{4}\right]^{1/2}\right\} \tag{3.2}$$

From this it is seen that q for all values of s is a purely imaginary quantity and the solution of the differential equation will be given by a cylindrical function of imaginary argument. Let us denote

$$r = \left\{\frac{a}{2} + \frac{\alpha^2}{4} + s^2 + \left[a\left(\frac{a}{2} + \frac{\alpha^2}{4} + s^2\right) - \frac{a^2}{4}\right]^{1/2}\right\}^{1/2} \tag{3.3}$$

A general solution for a cylinder extending infinitely from one end has the form

$$\varkappa(r, z) = \exp\left(-\frac{\alpha z}{2}\right)\int_0^\infty \{I_0(vr)\,[F_1(s)\cos(sz) + F_2(s)\sin(sz)] + \\ + K_0(vr)\,[F_3(s)\cos(sz) + F_4(s)\sin(sz)]\}\,ds \tag{3.4}$$

A general solution for a cylinder of a limited length l is

$$\varkappa(rz) = \exp\left(-\frac{\alpha z}{2}\right)\sum_{k=0}^{\infty}\left\{I_0(v_k r)\left[A_k\cos\frac{\pi k}{l}z + B_k\sin\frac{\pi k}{l}z\right] + \\ + K_0(v_k r)\left[C_k\cos\frac{\pi^k}{l}z + D_k\sin\frac{\pi^k}{l}z\right]\right\} \tag{3.5}$$

Here l is the length of the cylinder.

$$v_k = \left\{\frac{a}{4} + \frac{\alpha^2}{2} + \frac{\pi^2 k^2}{l^2} + \left[a\left(\frac{a}{2} + \frac{\alpha^2}{4} + \frac{\pi^2 k^2}{l^2}\right) - \frac{a^2}{4}\right]^{1/2}\right\}^{1/2} \qquad (k = 0, 1, 2, \ldots) \tag{3.6}$$

From (3.4) and (3.5) one can find the components of the stress tensor and displacement tensor which are suitable for the solution of boundary problems on the side surface of the infinite and finite cylinders.

Translated by **B. Z.**